# Price of Anarchy for the $N$-player Competitive Cascade Game with Submodular Activation Functions

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**Abstract.** We study the Price of Anarchy (PoA) of the competitive cascade game following the framework proposed by Goyal and Kearns in [11]. Our main insight is that a reduction to a Linear Threshold Model in a time-expanded graph establishes the submodularity of the social utility function. From this observation, we deduce that the game is a valid utility game, which in turn implies an upper bound of 2 on the (coarse) PoA. This cleaner understanding of the model yields a simpler proof of a much more general result than that established by Goyal and Kearns: for the $N$-player competitive cascade game, the (coarse) PoA is upper-bounded by 2 under any graph structure. We also show that this bound is tight.

**Keywords:** Competitive cascade game, Price of Anarchy, Submodularity, Valid utility game, Influence maximization

## 1 Introduction

The processes and dynamics by which information and behaviors spread through social networks have long interested scientists within many areas [18]. Understanding such processes has the potential to shed light on human social structure, and to impact the strategies used to promote behaviors or products. While the interest in the subject is long-standing, the recent increased availability of social network and information diffusion data (through sites such as Facebook and LinkedIn) has put into relief algorithmic questions within the area, and led to widespread interest in the topic within the computer science community.

One particular application that has been receiving interest in enterprises is to use word-of-mouth effects as a tool for viral marketing. Motivated by the marketing goal, mathematical formalizations of influence maximization have been proposed and extensively studied by many researchers [9, 14, 17, 23, 24, 8, 7, 16]. Influence maximization is the problem of selecting a small set of seed nodes in a social network, such that their overall influence on other nodes in the network — defined according to particular models of diffusion — is maximized.

When considering the word-of-mouth marketing application, it is natural to realize that multiple companies, political movements, or other organizations may use diffusion in a social network to promote their products simultaneously. For example, Samsung may try to promote their new Galaxy phone, while Apple tries to advertise their new iPhone. Companies will necessarily end up in competition with each other, so it becomes essential to understand the outcome of competitive diffusion phenomena in the network.

Motivated by the above scenarios, several models for competitive diffusion have been proposed and studied [2, 4, 7, 12, 20, 11, 1, 21]. Past work tends to follow one of two assumptions about the timing of players' moves. The first approach is to assume that all but one of the competitors have already chosen their strategies, and to study the algorithmic problem of finding the best response [2, 4, 7, 12, 5, 6]. The goal may be maximizing one's own influence [2, 4, 7] or minimizing the influence of the competitors [12,

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5]. The other approach is to model the competition as a simultaneous game, in which all companies pick their strategies at the same time [1, 11, 20, 21]. The final influence is determined by the initial seed set of every company and the underlying diffusion process.

In this paper, we follow the second approach. In the game, the players are companies (or other organizations) who try to promote their competing products in the social network through word-of-mouth marketing. The players simultaneously allocate resources to individuals in the social network in order to seed them as initial adopters of their products. These resources could be free samples, time spent explaining the advantages of the product, or monetary rewards. Based on the allocated resources, the nodes choose which of the products to adopt initially. Subsequently, the diffusion of the adoption of products proceeds according to the local adoption dynamics. The goal for each player is to maximize the coverage of his[1] own product.

The local adoption dynamics play a vital role in determining properties of the game. In this paper, we follow the framework proposed recently by Goyal and Kearns [11]. Their model decomposes the local adoption decisions into two stages: *switching* and *selection*. In the switching stage, the user decides whether to adopt any product or company at all. This decision is based on the set of neighbors who have already adopted one of the products. If the user decides to adopt a product, in the following selection stage, she decides which company's product to adopt based on the fraction of neighbors who have adopted the product from each company.

For example, assume that iPhone and Galaxy are the only two smartphones available. In the switching stage, a user decides whether to adopt a smartphone or not, based on the fraction of her neighbors who have already bought a smartphone. If she has decided to adopt a smartphone, in the selection stage, she decides whether to choose an iPhone or Galaxy based on the fraction of iPhone users and Galaxy users among her friends. The two stages are modeled using a *switching function* $f_v(\alpha_1 + \alpha_2)$, which gives the probability that the user adopts one of the products, and the *selection function* $g_v(\alpha_1, \alpha_2)$, which determines the probability that the user chooses the product of a specific company. Here $\alpha_1$ and $\alpha_2$ are the fractions of the user's friends who have already adopted the product from the two competitors. The details of the model are presented in Section 2.

Under this framework, Goyal and Kearns have studied the *Price of Anarchy* (PoA) of the two-player competitive cascade game. Informally, the PoA is a measure of the maximum potential inefficiency created by non-cooperative activity. (The precise definition of the PoA is given in Section 2.3.) Goyal and Kearns have shown that the PoA under the switching-selection model with concave switching functions and linear selection functions is upper-bounded by 4.[2]

In this paper, we show that a stronger PoA bound for the Goyal-Kearns model follows from several well-understood and general phenomena. The key observation is that by considering a time-expanded graph, the Goyal-Kearns model can be considered an instance of a general threshold model. Then, the result of Mossel and Roch [17] guarantees that the social utility function is submodular, and a simple coupling argument establishes that players' utility functions are competitive. With a submodular social utility function, the game is a valid utility game. (This type of proof was used previously by Bharathi et al. [2].) Finally, for valid utility games, the results of Vetta [22] and Blum et al. and Roughgarden [3, 19] establish a (coarse) Price of Anarchy of at most 2.

Thanks to the above understanding, we obtain a much more general result with a much simpler proof. We show that the PoA is upper-bounded by 2 for the competitive cascade game with an arbitrary number of players and any graph structure with submodular activation functions $f_v(\cdot)$. We formally state this result in Theorem 1. Moreover, by utilizing the result of Roughgarden in [19], we show that our bound not only holds for the PoA under pure or mixed Nash equilibria but also for the coarse PoA. We also show that the proposed PoA bound is tight.

[1] Throughout the paper, to simplify the distinction of roles, we consistently use "she" to denote individuals in the social network and "he" to denote the players, i.e., the companies.

[2] In fact, they proved that the PoA upper bound holds in a more general model, which we will discuss in Section 2.

**Theorem 1.** *The coarse PoA is upper-bounded by 2 under the switching-selection model with concave switching functions and linear selection functions.*[3]

Our result on the PoA bound holds under a generalized version of the framework used in [11]. First, and most importantly, our model allows for an arbitrary number of players. Second, we allow multiple players to target the same individual and allow each player to put multiple units of budget on the same individual.[4] This generalization enlarges the strategy space from sets to multisets and somewhat complicates the analysis of our model. Third, we associate each individual in the network with a weight measuring the importance of the node. Fourth, we generalize the adoption functions defined on the fraction of already adopting neighbors to arbitrary set functions defined on the individuals who have previously adopted the product.

### 1.1 Related Work

Our work is mainly motivated by [11], lying at the intersection of influence analysis in social networks and traditional game theory research. The model in [11] and the differences compared to our work are discussed in detail above and in Section 2.

Submodularity has been a recurring topic in the study of diffusion phenomena [17, 14, 12, 2, 4, 5]. [14, 17] have shown that influence coverage is submodular under local dynamics with submodularity. The submodularity of global influence coverage can be utilized to design efficient algorithm for either maximizing the influence [14] or minimizing the influence of the competitors [12]. Submodularity has also been applied in the analysis of a competitive influence game by Bharathi et al. [2]. Bharathi et al. use a similar approach as we do in this paper; they also bound the PoA bound by showing that the game is a valid utility game. However, they analyze the competitive cascade game under a simpler diffusion model. Under their model, a node adopts the product from the neighbor who first succeeds in activating her; a continuous timing component ensures that this node is unique with probability 1.

In the proof for the PoA bound of the competitive cascade game, we are drawing heavily on previous research on the PoA for valid utility games [22, 19, 3]. Vetta first showed that for a valid utility game, the PoA for pure Nash equilibria is upper-bounded by 2 in [22]. Blum et al. and Roughgarden later generalized Vetta's result to the coarse PoA in [19, 3].

Several other game-theoretic approaches have been considered for competitive diffusion in social networks [21, 1, 20, 10, 6]. [20] mainly focuses on the efficient computation of the Nash strategy instead of the theoretical bound of the PoA. [6] focuses on studying the algorithmic problem of finding the best response. Though [1, 21, 10] studied the competitive cascade game from a game-theoretic perspective, they mainly focused on the existence of pure Nash equilibria. [1] mainly focuses on the existence of pure Nash equilibria under a deterministic threshold model. [10] also tries to characterize the structure of the pure Nash equilibria in the game. The PoA is studied in [21]; however, they studied the PoA bound of pure Nash equilibria and used a deterministic diffusion model instead of the stochastic dynamics we use in our work. In their model, the PoA is unbounded as in the Goyal-Kearns model with non-concave switching functions. As noted by [4, 5], small differences in the diffusion model can lead to dramatically different behaviors of the model.

## 2 Models and Preliminaries

In this section, we define basic notation, present the different models of diffusion and the $N$-player game, and include other definitions of concepts used in our proof. In the game, the players allocate resources to

[3] Similar to the result by Goyal and Kearns, our PoA upper bound extends to a more general model. We define this more general model in Section 2, and state and prove the more general result in Section 3.

[4] The model proposed by Goyal and Kearns [11] allows for multiple units of budget on the same individual, but the proof does not explicitly cover this extension.

the nodes in the graph $G = (V, E)$ to win them as initial adopters of their products. Then, the adoption of products propagates according to the local dynamics, described in detail in Section 2.1. The formal definition of the game is presented in Section 2.3.

Throughout, we use the following conventions for notation. Players are typically denoted by $i, j, k$, while nodes are $u, v, w$. For sets, functions, etc., the identity of a player is applied as a superscript, while that of a node (and time step) is applied as a subscript. Vectors are written in boldface, including vectors of sets; in particular, we frequently write $\boldsymbol{S} = (S^1, \ldots, S^N)$ for the vector of sets of nodes belonging to the different players.

### 2.1 General adoption model

The *general adoption model* is a generalization of the switching-selection model described in Section 1. Each node in $G$ is in one of the $N + 1$ states $\{0, 1, \ldots, N\}$. A node $v$ in state $i > 0$ means that individual $v$ has adopted the product of player $i$, while state 0 means that she has not adopted the product of any player. In this case, we also say that $v$ is *inactive*. Conversely, we say that node $v$ is *activated* if she is in one of the states $i > 0$. Initially, all nodes are inactive. The diffusion of the adoption of products is a process described by nodes' state changes. We assume that the process is *progressive*, meaning that a node can change her state at most once, from 0 to some $i > 0$, and must remain in that state subsequently.

The diffusion process works in two stages. We call the first stage *Seeding* and the second stage *Diffusion*. In the first stage, the initial seeds of all players are decided based on the budgets that each player allocates to the nodes. The initial seeds are used as starting points for the diffusion stage. In the second stage, the adoption propagates according to certain local dynamics based on the nodes who have previously adopted the products.

*Seeding stage:* The strategy $M^i$ of player $i$ is a multiset of nodes. We define $\alpha_v^i$ as the number of times that $v$ appears in player $i$'s multiset. For each node $v \in V$, if $\sum_{i=1}^{N} \alpha_v^i = 0$, the initial state of node $v$ is 0; otherwise, the initial state of node $v$ is one of $\{1, 2, \ldots, N\}$ with probabilities $(\frac{\alpha_v^1}{Z_v}, \ldots, \frac{\alpha_v^N}{Z_v})$, where $Z_v = \sum_{i=1}^{N} \alpha_v^i$ is simply the normalizing constant. The decisions for different nodes are independent. Thus, if no player selects a node, the node remains inactive. Otherwise, the players win the node as an initial adopter with probability proportional to the number of times they select the node.

*Diffusion stage:* The important part of diffusion is the local dynamics deciding when a node gets influenced, i.e., changes her state from 0 to $i$. Let $S^i$ be the set of nodes in state $i$. A node $v$ who is still in state 0 changes into state $1, \ldots, N, 0$ according to the probabilities

$$(h_v^1(\boldsymbol{S}), \ldots, h_v^N(\boldsymbol{S}), 1 - \sum_{i=1}^{N} h_v^i(\boldsymbol{S})).$$

We call $h_v^i(S^1, \ldots, S^N)$ the *adoption function* of node $v$ for product $i$. It gives the probability that a still inactive node $v$ adopts product $i$ given that $S^j$ is the current set of nodes in state $j$. The adoption functions must satisfy the following two conditions:

$$0 \leq h_v^i(\boldsymbol{S}) \leq 1, \quad \forall v \in V, i = 1, \ldots, N$$
$$\textstyle\sum_{i=1}^{N} h_v^i(\boldsymbol{S}) \leq 1, \forall v \in V.$$

We call $H_v(\boldsymbol{S}) = \sum_{i=1}^{N} h_v^i(\boldsymbol{S})$ the *activation probability*; it gives the probability that $v$ adopts any product and changes from state 0 to any state $i > 0$.

Equipped with the local dynamics of adoption, we still need to define in what order nodes' states are updated. In the general adoption model, we assume that an update schedule is given in advance to determine the order of updates. The update schedule is a finite sequence $Q$ of nodes $\langle v_1, \ldots, v_\ell \rangle$, of length $\ell$. A node could occur multiple times in the sequence.

Nodes' states are updated according to the order prescribed by the sequence. Let $S_t^i$ be the set of nodes in state $i$ after the first $t$ updates; $S_0^i$ is the seed set of player $i$ resulting from the seeding stage. In each round $t$, the state of node $v_t$ is updated according to the local dynamics of adoption and previously activated nodes, namely $\boldsymbol{S}_{t-1} = (S_{t-1}^1, \ldots, S_{t-1}^N)$. If node $v_t$ is already in state $i > 0$, she remains in state $i$. Otherwise, she changes into state $1, \ldots, N, 0$ according to the probabilities

$$(h_v^1(\boldsymbol{S}_{t-1}), \ldots, h_v^N(\boldsymbol{S}_{t-1}), 1 - \sum_{i=1}^{N} h_v^i(\boldsymbol{S}_{t-1})).$$

The states of all other nodes remain the same. The updates in different rounds are independent. The diffusion stage ends after the $\ell$ update steps. The prescribed update sequence makes this model different from the previously studied Independent Cascade and Threshold Models. We discuss the difference and some implications in more detail after defining the Threshold Model in Section 2.4.

### 2.2 Useful properties

We next identify three important properties that make the model more tractable analytically: (1) additivity of the activation probability $H_v$, (2) competitiveness of the adoption function $h_v$ and (3) submodularity of the activation function $f_v$.

**Definition 1.** *The total activation probability $H_v(\boldsymbol{S}) = \sum_{i=1}^{N} h_v(\boldsymbol{S})$ is* additive *if and only if $H_v$ can be written as $H_v(\boldsymbol{S}) = f_v(\bigcup_{i=1}^{N} S^i)$ for some monotone set function defined on $V$. We call $f_v(S)$ the* activation function *for $v$ when $H_v$ is additive.*

Additivity implies that the probability for a node to adopt the product and change from inactive to active only depends on the set of already activated nodes and not on which specific products they have adopted. For example, the probability that one adopts a smartphone only depends on who has already adopted one, independent of who is using iPhone and who is using Galaxy.

To simplify notation, we define $S^{-i} = \bigcup_{k \neq i} S^k$, and $\boldsymbol{S}^{-i} = (S^k)_{k \neq i}$.

**Definition 2.** *The adoption function $h_v^i(\boldsymbol{S})$ for player $i$ is* competitive *if and only $h_v^i(\boldsymbol{S}) \geq h_v^i(\hat{\boldsymbol{S}})$ whenever $\hat{S}^i \subseteq S^i$ and $S^{-i} \subseteq \hat{S}^{-i}$.*

Competitiveness means that the adoption function for player $i$ is monotone increasing in the set of nodes that have adopted product $i$ and monotone decreasing in the set of nodes that have adopted some competitor's products.[5]

**Definition 3.** *The activation function $f_v$ is* submodular *if and only if for any two set $S \subseteq T \subseteq V$ and any node $u \in V$,*

$$f_v(S \cup \{u\}) - f_v(S) \geq f_v(T \cup \{u\}) - f_v(T).$$

[5] This assumption is reasonable when the reputation of the product is already well-established. However, when a new product comes out, the presence of competitors may help popularize the product, by increasing its overall exposure or perceived importance or relevance. These effects could lead to more purchases even for one particular company $i$. This subtle distinction is discussed more in Section 2.4.

Submodularity of activation functions implies that the overall activation probability has diminishing returns. It intuitively means that the first friend to buy and recommend a smartphone has more influence than a friend who recommends it after many others.

Goyal and Kearns have shown in [11] that the switching-selection model with concave switching functions and linear selection functions is a special case of the general adoption model with competitive adoption functions and additive activation probabilities. In addition, due to the concavity of the switching function $f_v$, the activation functions in the general adoption model are also submodular. Therefore, we have the following lemma:

**Lemma 1.** *Every instance of the switching-selection model with concave switching functions and linear selection functions is an instance of the general adoption model satisfying all three of the above properties.*

Lemma 1 allows us to prove our PoA bound only for the general adoption model; it then implies Theorem 1.

### 2.3 The game

The competitive cascade game is an $N$-player game on a given graph $G = (V, E)$. The structure of the graph as well as all adoption functions are known to all the players. Each player $i$ is a company. The strategy for each player $i$ is a multiset $M^i$ of nodes; we use $\boldsymbol{M} = (M^1, \ldots, M^N)$ to denote the strategy vector for all players and $\alpha_v^i$ for the number of times that node $v$ appears in $M^i$.

Players' strategies are constrained by their budgets $B^i$, in that they must satisfy $|M^i| \leq B^i$. We further allow node-specific constraints requiring that $\alpha_v^i \leq K_v^i$ for given node-specific budgets $K_v^i \leq B^i$. These may constrain players from investing a lot of resources into particularly hard-to-reach nodes; however, the node-specific constraints mostly serve to simplify notation in some later proofs. We say that a strategy $M^i$ is *feasible* if all of the above conditions are satisfied.

All players simultaneously allocate their budgets to the nodes of $G$. Given the choices that the players make, the payoffs are determined by the general adoption model as the coverage of the player's product among the individuals in $G$. Each node $v$ in the graph is associated with a weight $\omega_v \geq 0$, measuring the importance of the node. The payoff function of player $i$ is $\sigma^i(\boldsymbol{M}) = \mathbb{E}[\sum_{v \in S_\ell^i} \omega_v]$, the expected sum of weights from nodes having adopted $i$'s product after all $\ell$ update steps.

The social utility $\gamma(S_0) = \sum_i \sigma^i(\boldsymbol{M})$ is the sum of weights from nodes adopting any of the products.[6] Notice that when the activation probabilities $H_v$ are additive (Definition 1), $\gamma(\cdot)$ only depends on $S_0 = \bigcup_{i=1}^N S_0^i$, the set of nodes activated after the seeding stage (but not on which company they chose).

To simplify notation, we define $(\boldsymbol{M}^{-k}, \tilde{M}^k) = (M^1, \ldots, M^{k-1}, \tilde{M}^k, M^{k+1}, \ldots, M^N)$, and in particular $(\boldsymbol{M}^{-k}, \emptyset^k) = (M^1, \ldots, M^{k-1}, \emptyset, M^{k+1}, \ldots, M^N)$.

We say that a strategy profile $\boldsymbol{M}$ is a *pure Nash equilibrium* if no player has an incentive to change his strategy. Namely, for any player $i$,

$$\sigma^i(\boldsymbol{M}) \geq \sigma^i(\boldsymbol{M}^{-i}, \tilde{M}^i) \quad \text{for all feasible } \tilde{M}^i.$$

Let $\mathbf{OPT}$ be a strategy profile maximizing the social utility function, and $\mathrm{EQ}_{\mathrm{pure}}$ the set of all pure Nash equilibria. The price of anarchy of pure Nash equilibria is defined as follows:

$$\text{Pure Price of Anarchy} = \max_{\boldsymbol{M} \in \mathrm{EQ}_{\mathrm{pure}}} \frac{\gamma(\mathbf{OPT})}{\gamma(\boldsymbol{M})}.$$

[6] This definition implicitly assumes that the product carries a value for those who adopt it; thus, society is better off when more people adopt at least one product.

However, the competitive cascade game could have no pure Nash equilibrium [21]. Thus, we extend our analysis to more general equilibrium concepts. A *coarse (correlated) equilibrium* of a game is a joint probability distribution $\mathbf{P}$ with the following property [19]: if $\boldsymbol{M}$ is a random variable with distribution $\mathbf{P}$, then for each player $i$, and all feasible $\hat{M}^i$:

$$\mathbb{E}_{\boldsymbol{M}\sim\mathbf{P}}[\sigma^i(\boldsymbol{M})] \geq \mathbb{E}_{\boldsymbol{M}^{-i}\sim\mathbf{P}^{-i}}[\sigma^i(\boldsymbol{M}^{-i}, \hat{M}^i)].$$

Similar to the PoA for pure Nash equilibria, the *coarse price of anarchy* is defined as

$$\text{Coarse Price of Anarchy} = \max_{\mathbf{P}\in\mathrm{EQ}_{\text{coarse}}} \frac{\gamma(\mathbf{OPT})}{\mathbb{E}_{\boldsymbol{M}\sim\mathbf{P}}\gamma(\boldsymbol{M})},$$

where $\mathrm{EQ}_{\text{coarse}}$ is the set of all coarse equilibria.

### 2.4 The Threshold Model

Our analysis will be based on a careful reduction of the general adoption model to the general threshold model defined in [14, 15]. In the general threshold model (with $N = 1$), every node $v$ in the network has an associated activation function $\hat{f}_v(\cdot)$. At the beginning of the process, each node draws a threshold $\theta_v$ independently and uniformly from $[0, 1]$. Starting from an initially active set $S_0$, a node becomes active at time $t$ (i.e., is a member of $S_t$) if and only if $\hat{f}_v(S_{t-1}) \geq \theta_v$. The process ends when for one round, no new node has become active (which is guaranteed to happen in at most $|V|$ steps). If $t$ is the time when this happens, the influence of the initial set $S_0$ is defined as $\sigma_\omega(S_0) = \mathbb{E}[\sum_{v\in S_t} \omega_v]$. In a beautiful piece of work, Mossel and Roch established the following theorem about the function $\sigma_\omega$:

**Theorem 2 (Mossel-Roch [17]).** *If $f_v$ is monotone and submodular for every node $v$ in the graph, then $\sigma_\omega$ is monotone and submodular under the general threshold model.*

Given the apparent similarity between the general adoption model and the general threshold model (say, for $N = 1$), it is illuminating to consider the ways in which the models differ, and the implications for the competitive cascade game. In the general adoption model, a sequence of nodes to update is given, and nodes only consider changing their state when they appear in the sequence. By contrast, in the general threshold model, nodes consider changing their state in each round.

So at first, it appears as though a sequence repeating $|V|$ times a permutation of all $|V|$ nodes would allow a reduction from the threshold model to the adoption model. However, note that in the adoption model, each node makes an *independent* random choice whether to change her state in each round, whereas in the threshold model, the random choices are coupled via the threshold $\theta_v$, which stays constant throughout the process. In particular, if $f_v(S_0) > 0$, then a node appearing often enough in the update sequence will eventually be activated with probability converging to 1, whereas this need not be the case in the general threshold model.

The increase in activation probability caused by multiple occurrences in the update sequence has powerful implications for the competitive game. It allows us to establish rather straightforwardly the competitiveness (Definition 2) of each player's objective function, and the submodularity (Definition 3) of the social utility. By contrast, Borodin et al. [4] show that both properties fail to hold for most natural definitions of competitive threshold games. At the heart of the counter-examples in [4] lies the following kind of dynamic: At time 1, a node $u$ recommends to $v$ the use of a Galaxy phone, but fails to convince $v$. At time 2, another node $w$ recommends to $v$ the use of an iPhone. If $v$ decides to adopt a smartphone at time 2, most natural versions of a threshold model (as well as under the general adoption model) allow for an adoption of a Galaxy phone as well. This "extra chance" results in synergistic effects between competitors, and thus breaks competitiveness. Under the model of [11], this problem is side-stepped. $v$ will only consider adopting a smartphone in step 2 when she appears in the sequence at time 2; in that case, adoption of a Galaxy phone in step 2 will be considered independently of whether $w$ has adopted an iPhone. This observation fleshes out the discussion alluded to in Footnote 5.

### 2.5 Valid utility games

A valid utility game [22] is defined on a ground set $V$ with social utility function $\gamma$ defined on subsets of $V$. The strategies of the game are sets $S_0^i \subseteq V$ (it is possible that not all sets are allowed as strategies for some or all players), and the payoff functions are $\sigma^i$ for each player $i$. The social utility is defined on the union of all players' sets, $\gamma(\bigcup_{i=1}^N S_0^i)$. The definition requires that three conditions hold: (1) The social utility function $\gamma(\cdot)$ is submodular; (2) For each player $i$, $\sigma^i(\boldsymbol{S_0}) \geq \gamma(\boldsymbol{S_0}) - \gamma(\boldsymbol{S_0}^{-i}, \emptyset^i)$; (3) $\sum_{i=1}^N \sigma^i(\boldsymbol{S_0}) \leq \gamma(\boldsymbol{S_0})$.

## 3 Upper Bound on the Coarse Price of Anarchy

In this section, we present our main result: the upper bound on the coarse PoA with submodular activation functions. We prove the upper bound on the PoA by showing that the competitive cascade game is a valid utility game. We note, however, that the strategy space of our competitive cascade game consists of multisets, whereas the standard definition of utility games has only sets as strategies. In order to deal with this subtle technical issue, we use the following lemma, whose proof is given in the appendix.

**Lemma 2.** *Let $\mathcal{G} = (G, \{h_v^i\}, \{K_v^i\}, \{B^i\}, \{\omega_v\}, Q)$ be an arbitrary instance of the competitive cascade game. Then, there exists an instance $\hat{\mathcal{G}} = (\hat{G}, \{\hat{h}_v^i\}, \{\hat{K}_{\hat{v}}^i\}, \{\hat{B}^i\}, \{\hat{\omega}_v\}, \hat{Q})$ with the same set of players, and the following properties:*

1. *$\sum_{i=1}^N \hat{K}_{\hat{v}}^i \leq 1$ for all $v \in \hat{V}$. (At most one player is allowed to target a node, and with at most one resource.)*
2. *For every player $i$, there are mappings $\mu^i, \hat{\mu}^i$ mapping $i$'s strategies in $\mathcal{G}$ to his strategies in $\hat{\mathcal{G}}$ and vice versa, respectively, satisfying the following property: If for all $i$, either $\mu^i(M^i) = \hat{M}^i$ or $\hat{\mu}^i(\hat{M}^i) = M^i$, then for all $i$, $\sigma^i(\boldsymbol{M}) = \hat{\sigma}^i(\hat{\boldsymbol{M}})$.*

In particular, Lemma 2 implies that the social utility is also preserved between the two games, and strategies $M^i$ are best responses to $M^j, j \neq i$ if and only if the $\hat{M}^i$ are best response to $\hat{M}^j, j \neq i$. (Otherwise, a player could improve his payoff in the other game by switching to $\mu^i(M^i)$ or $\hat{\mu}^i(\hat{M}^i)$.) In this sense, Lemma 2 establishes that for every competitive cascade game instance, there is an "equivalent" instance in which each node can be targeted by at most one player, and with at most one resource. Therefore, we will henceforth assume without loss of generality that the strategy space for each player consists only of sets.

In fitting the competitive cascade game into the valid utility game framework, the ground set of the game is $V$, and the payoff function of player $i$ is $\sigma^i(\boldsymbol{S}) = \mathbb{E}[\sum_{v \in S_\ell^i} \omega_v]$: the sum of weights from the nodes $v \in V$ in state $i$ at the end of the updating sequence. Because of additivity, the social utility function depends only on $S_0$. That is, the following is well-defined: $\gamma(\boldsymbol{S_0}) = \gamma(S_0) = \sum_i \sigma^i(\boldsymbol{S_0})$. Therefore, the third condition of a valid utility game (sum boundedness) is satisfied trivially. Below, we will prove the following two lemmas:

**Lemma 3.** *Assume that for every node $v$, the total activation probability $H_v(\boldsymbol{S})$ is additive, and the activation function $f_v(S)$ is submodular. Then, the social utility function $\gamma(S_0)$ is submodular and monotone.*

**Lemma 4.** *If for every node $v$ and ever player $k$, $H_v(\cdot)$ is additive and $h_v^k(\cdot)$ is competitive, then for each player $i$, we have $\sigma^i(\boldsymbol{S_0}) \geq \gamma(\boldsymbol{S_0}) - \gamma(\boldsymbol{S_0}^{-i}, \emptyset^i)$.*

Lemmas 3 and 4 together establish that the competitive cascade game is a valid utility game. Example 1.4 in [19] shows that the coarse PoA of valid utility games is at most 2 (Vetta [22] establishes the same for the PoA), proving the following main result of our paper:

**Theorem 3.** *Assume that the following conditions hold:*

1. *For every node $v$, the total activation probability $H_v(\boldsymbol{S})$ is additive.*
2. *For every node $v$, the activation function $f_v(S)$ is submodular.*
3. *For every player $i$ and node $v$ in the graph, the adoption function $h_v^i(\boldsymbol{S})$ is competitive.*

*Then, the upper bound on the PoA (and coarse PoA) is 2 in the competitive cascade game.*

By Lemma 1, the switching-selection model with concave switching functions and linear selection functions is a special case of the general adoption model with competitive adoption functions, additive activation probabilities and submodular activation functions. Therefore, Theorem 1 follows naturally as a corollary of Theorem 3.

**Proof of Lemma 3.** We build an instance of the general threshold model whose influence coverage function $\sigma_\omega(S_0)$ is exactly the same as $\gamma(S_0)$. The idea is that for additive functions, the social utility does not depend on which node chooses which company, so the game is reduced to the case of just a single influence. The update sequence can be emulated with a time-expanded layered graph.

The time-expanded graph $G_\ell$ is defined as follows.[7] For each node $v$ of the original graph, we have $\ell + 1$ nodes $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_\ell$ in $G_\ell$. We use $L_t = \{\hat{v}_t \mid v \in V\}$ to denote the set of nodes in layer $t$. The activation functions are defined as follows:

1. In layer 0, $\hat{f}_{\hat{v}_0} \equiv 0$ for every node $v \in V$.
2. In layer $t$, $1 \leq t \leq \ell$, consider a node $v$ with switching function $f_v$. If $v$ is the $t^{\text{th}}$ element of the updating sequence ($v = v_t$), we set

$$\hat{f}_{\hat{v}_t}(S) = \begin{cases} 1 & \text{if } \hat{v}_{t-1} \in S \\ f_v(\{u \mid \hat{u}_{t-1} \in S\}) & \text{otherwise;} \end{cases}$$

otherwise we set

$$\hat{f}_{\hat{v}_t}(S) = \begin{cases} 1 & \text{if } \hat{v}_{t-1} \in S \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the total influence is defined as $\sigma_\omega(S_0) = \mathbb{E}[\sum_{\hat{v}_\ell \in \hat{S}} \omega_v]$, where $\hat{S}$ is the set of nodes activated in the threshold model once no more activations occur. In the instance, each layer $L_t$ emulates one update in the original update sequence of the general adoption model.

For each node $\hat{v}_t$ in the layered graph, $f_{\hat{v}_t}(S)$ is submodular and additive. The submodularity and monotonicity for the 0-1 activation functions are trivially satisfied. For the nodes in the update sequence, submodularity holds because we assumed the $f_v(S)$ to be submodular, and monotonicity follows because the $H_v(\boldsymbol{S})$ are additive.

Next, we show that $\gamma(S_0) = \sigma_\omega(S_0)$, by using a straightforward coupling between the general threshold model and the general adoption model. According to the construction of $G_\ell$, the state changes for all nodes except $\hat{v}_t$ (where $v_t$ is the $t^{\text{th}}$ element of the updating sequence) are deterministic. Therefore, we only need to draw the thresholds $\Theta = \langle \theta_1, \theta_2, \ldots, \theta_\ell \rangle$ for the $\ell$ nodes in the update sequence: they are drawn independently and uniformly from $[0, 1]$. In the general adoption model, when updating the $t^{\text{th}}$ node $v_t$ in the sequence, if node $v_t$ is still inactive, she becomes active if and only if $f_v(S_{t-1}) \geq \theta_t$. If the node is already activated, she remains activated in the same state. By induction on $t$, $v \in S_t$ if and only if $\hat{v}_t \in \hat{S} \cap L_t$. Thus, the outcomes of the two processes are the same pointwise over threshold vectors $\Theta$: $\gamma(S_0|\Theta) = \sigma_\omega(S_0|\Theta)$. In particular, their expectations are thus the same.

Finally, Theorem 2 establishes the monotonicity and submodularity of $\sigma_\omega(S_0)$, and thus also $\gamma(S_0)$. ■

[7] All activation information is encoded in the activation functions. Therefore, there is no need to explicitly specify the edges of $G_\ell$.

**Proof of Lemma 4.** We begin by showing that under the assumptions of the lemma, $\sigma^i(\boldsymbol{S_0}) \leq \sigma^i(\boldsymbol{S_0}^{-k}, \emptyset^k)$ for all players $k, i, k \neq i$. To do so, we exhibit a simple coupling of the general adoption processes for the two initial states $\boldsymbol{S_0}$ and $(\boldsymbol{S_0}^{-k}, \emptyset^k)$, essentially identical to one used in the proof of Lemma 1 in [11]. Notice that the activation functions are additive; therefore, we can combine all states $k \neq i$ into one state, which we denote by $-i$.

The activation process is defined by the way in which nodes decide whether to update their state, and if so, to which new state. An equivalent way of describing the choice is as follows: for each step $t$ of the update sequence, we draw an independent uniformly random number $z_t \in [0, 1]$. In step $t$, assuming that node $v_t$ is still in state 0, she changes her state to:

- state $i$ if $z_t \in [0, h^i_{v_t}(\boldsymbol{S}_{t-1}))$.
- state $-i$ if $z_t \in [h^i_{v_t}(\boldsymbol{S}_{t-1}), f_{v_t}(\bigcup_{j=1}^N S^j_{t-1}))$.
- state 0 otherwise.

To couple the two random processes with starting conditions $(\boldsymbol{S_0}^{-k}, \emptyset^k)$ and $\boldsymbol{S_0}$, we simply choose the same values $z_t$ for both. Let $X^j_t$ denote the set of nodes in state $j$, $j \in \{i, -i, 0\}$ after $t$ updates with starting condition $(\boldsymbol{S_0}^{-k}, \emptyset^k)$. $Y^j_t$ is defined analogously, with starting condition $\boldsymbol{S_0}$.

Conditioned on any choice of $(z_1, \ldots, z_\ell)$, a simple induction proof using competitiveness of the $h^i_v$ and monotonicity of the $f_v$ shows that for each time $t$, $X^i_t \supseteq Y^i_t$, $X^0_t \supseteq Y^0_t$, and thus also $X^{-i}_t \subseteq Y^{-i}_t$. Therefore, at the end of the update sequence, the desired inequality holds pointwise over $(z_1, \ldots, z_\ell)$, and in particular in expectation, as claimed. Finally, having established that $\sigma^i(\boldsymbol{S_0}) \leq \sigma^i(\boldsymbol{S_0}^{-k}, \emptyset^k)$, we use it in the following calculations:

$$\begin{aligned}
\gamma(\boldsymbol{S_0}) - \gamma(\boldsymbol{S_0}^{-i}, \emptyset^i) &= \sum_k (\sigma^k(\boldsymbol{S_0}) - \sigma^k(\boldsymbol{S_0}^{-i}, \emptyset^i)) \\
&= \sigma^i(\boldsymbol{S_0}) + \sum_{k \neq i} (\sigma^k(\boldsymbol{S_0}) - \sigma^k(\boldsymbol{S_0}^{-i}, \emptyset^i)) \\
&\leq \sigma^i(\boldsymbol{S_0}).
\end{aligned}$$

■

## 4 Tightness of the PoA Upper Bound

We give an instance of the competitive cascade game in the (more restrictive) switching-selection model to show that our upper bound of 2 for the PoA in Theorem 3 is tight.

Let $N$ be the number of players. The graph consists of a star with one center and $N$ leaves, as well as $N$ isolated nodes. Each player has only one unit of budget, and the update sequence is any permutation of the nodes in the star graph. The switching functions are the constant 1 for all nodes in the graph, which implies that if a node has any neighbor who has adopted the product, the node also adopts the product. The selection functions are simply the fraction of neighbors who have adopted the product previously. Under this instance, the unique Nash equilibrium has every player allocating his unit of budget to the center node of the star graph. By placing the budget at the center node, the expected payoff for each player is $\frac{N+1}{N}$, while placing it on any other node at most leads to a payoff of 1. However, the strategy that optimizes the social utility is to place one unit of budget at the center node of the star graph while placing all others at the isolated nodes. Thus, the PoA (and also Price of Stability) is $\frac{2N}{N+1}$. As $N$ goes to infinity, the lower bound on the PoA tends to 2. Therefore, we have proved the following proposition:

**Proposition 1.** *The upper bound of 2 on the PoA (and thus also coarse PoA) is tight for the competitive cascade game even for the simpler switching-selection model.*

## 5 Conclusion and Future Work

We have studied the efficiency of resource allocation at equilibria of the competitive cascade game in terms of the *Price of Anarchy* (PoA). We have shown that an improved bound compared to [11] follows from several well-understood and general phenomena. This cleaner approach has led to a simpler proof of a more general result: for the $N$-player competitive cascade game, the coarse PoA is upper-bounded by 2 under any graph structure. We have also shown that this bound is tight.

It is open whether the same (or a slightly weaker) bound can be guaranteed without the assumption of submodularity of the activation functions (but assuming competitiveness and additivity). The techniques from [11] can be generalized to give an upper bound of $2N$ in this case, but do not directly yield any better bounds.

At a more fundamental level, it would be desirable to broaden the models considered for competitive cascades. Most positive results on either algorithmic questions or the PoA — the present one included — rely on submodularity properties of the particular modeling choices. (That such properties are also at the heart of the model of Goyal and Kearns is our main insight here.) It would be desirable to find models for which positive results — algorithmic or game-theoretic — can be obtained without requiring submodularity. Furthermore, most work on cascade models so far has assumed that nodes only adopt a single product. In many cases, products may be *partly* in competition, but not fully so. One of the few papers to consider a model with partial compatibility between products is [13]; an exploration of the game-theoretic implications of such a model would be of interest.

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## A Proof of Lemma 2

We restate Lemma 2 for convenience.

**Lemma 2** *Let $\mathcal{G} = (G, \{h_v^i\}, \{K_v^i\}, \{B^i\}, \{\omega_v\}, Q)$ be an arbitrary instance of the competitive cascade game. Then, there exists an instance $\hat{\mathcal{G}} = (\hat{G}, \{\hat{h}_v^i\}, \{\hat{K}_{\hat{v}}^i\}, \{\hat{B}^i\}, \{\hat{\omega}_v\}, \hat{Q})$ with the same set of players, and the following properties:*

1. *$\sum_{i=1}^N \hat{K}_{\hat{v}}^i \leq 1$ for all $v \in \hat{V}$. (At most one player is allowed to target a node, and with at most one resource.)*
2. *For every player $i$, there are mappings $\mu^i, \hat{\mu}^i$ mapping $i$'s strategies in $\mathcal{G}$ to his strategies in $\hat{\mathcal{G}}$ and vice versa, respectively, satisfying the following property: If for all $i$, either $\mu^i(M^i) = \hat{M}^i$ or $\hat{\mu}^i(\hat{M}^i) = M^i$, then for all $i$, $\sigma^i(\boldsymbol{M}) = \hat{\sigma}^i(\hat{\boldsymbol{M}})$.*

**Proof.** Given $\mathcal{G}$, we construct $\hat{\mathcal{G}}$ as a game on a graph with three layers.

**Nodes:** The first layer contains, for each node $v \in V$ and player $i$, a set of $K_v^i$ new nodes $V_v^i = \{v_1, \ldots, v_{K_v^i}\}$. The second layer of $\hat{G}$ contains, for each node $v \in V$, one node $v'$ connected to all nodes in $V_v^i$. The third layer is a copy of the original graph $G$.

**Node Budgets:** For player $i$ and any node $v \in G$, we set $\hat{K}_{\hat{v}}^i = 1$ for all nodes $\hat{v} \in V_v^i$, and $\hat{K}_{\hat{v}}^i = 0$ for all other nodes (including all nodes in layers 2 and 3). In other words, player $i$ may only target nodes that are in $V_v^i$ for some $v \in G$.

**Budgets:** We set $\hat{B}^i = B^i$, for all players $i$.

**Weights:** We set $\hat{\omega}_v \equiv 0$ for all nodes in the first layer. If a node $v$ appears (at least once) in $Q$, then we set $\hat{\omega}_v = \omega_v$ in the third layer and $\hat{\omega}_{v'} = 0$ in the second layer. If $v$ does not appear in $Q$, then we set $\hat{\omega}_v = 0$ in the third layer and $\hat{\omega}_{v'} = \omega_v$ in the second layer. Thus, players are interested in influencing nodes in the second or third layer, depending on whether the node can be influenced via the update sequence, or must be influenced by direct targeting.

**Adoption Functions:** Different adoption functions are used for the nodes in different layers:

1. In layer 1, $\hat{h}_{\hat{v}}^i(\cdot) \equiv 0$ for any player $i$ and node $\hat{v}$.
2. In layer 2, for a player $i$ and node $v$,

$$\hat{h}_{v'}^i(\boldsymbol{S}) = \begin{cases} 0 & \text{if } \bigcup_{k=1}^N (S^k \cap V_v^k) = \emptyset \\ \frac{|S^i \cap V_v^i|}{\sum_k |S^k \cap V_v^k|} & \text{otherwise.} \end{cases}$$

3. To simplify notation, we define $A^i = \{v \in V | v \in S^i \text{ or } v' \in S^i\}$. Then in layer 3, for player $i$ and node $v$,

$$\hat{h}_v^i(\boldsymbol{S}) = \begin{cases} 0 & \text{if } v' \in S^j \text{ for some } j \neq i \\ 1 & \text{if } v' \in S^i \\ h_v^i(A^1, \ldots, A^N) & \text{otherwise.} \end{cases}$$

Notice that the game $\hat{\mathcal{G}}$ satisfies competitiveness, additivity and submodularity whenever the game $\mathcal{G}$ satisfies all these three properties.

**Update Sequence:** The update sequence is $\hat{Q} = \langle v_1', v_2', \ldots, v_{|V|}', v_1, \ldots, v_\ell \rangle$, where $Q = \langle v_1, \ldots, v_\ell \rangle$ is the update sequence of the original instance and $v_1', v_2', \ldots, v_{|V|}'$ are all the nodes in the second layer, in some arbitrary order. The first $|V|$ updates in $\hat{\mathcal{G}}$ emulate the seeding stage in $\mathcal{G}$, and the remaining $\ell$ updates emulate the update sequence $Q$.

**Payoffs and Social Utility:** The players' payoff functions $\hat{\sigma}^i(\boldsymbol{M})$ and the social utility $\hat{\gamma}(\boldsymbol{M})$ are defined as usual in terms of the other modeling parameters.

The mappings $\mu^i$ are defined as follows. Let $M^i$ be $i$'s strategy in $\mathcal{G}$, characterized by the budgets $\alpha_v^i$ that $i$ puts on nodes $v$. For each node $v \in V$, we choose an arbitrary (but fixed) set $\hat{M}_v^i$ of $\alpha_v^i$ nodes in $V_v^i$. Player $i$'s strategy is $\hat{M}^i = \hat{\mu}^i(M^i) = \bigcup_v \hat{M}_v^i$. Conversely, we define $\hat{\mu}^i$ as follows: For any strategy profile $\hat{\boldsymbol{M}}$ of $\hat{\mathcal{G}}$, and for each node $v \in V$, we set $\alpha_v^i = |\hat{M}^i \cap V_v^i|$. $\hat{\mu}^i(\hat{M}^i)$ is the strategy in which player $i$ puts $\alpha_v^i$ resources on node $v$, for all $v$.

The first claim of the lemma holds by definition. For the second claim, consider two strategy profiles $\boldsymbol{M}$ and $\hat{\boldsymbol{M}}$, such that for all players $i$, either $\hat{M}^i = \mu^i(M^i)$ or $M^i = \hat{\mu}^i(\hat{M}^i)$. We show that $\hat{\sigma}^i(\hat{\boldsymbol{M}}) = \sigma^i(\boldsymbol{M})$ for each player $i$. To do so, we exhibit a coupling of the random choices between the two games $\mathcal{G}$ and $\hat{\mathcal{G}}$. The coupling is quite similar to the one used in the proof of Lemma 4.

For the seeding stage of $\mathcal{G}$, an equivalent way of describing the choice is as follows: for each node $v \in V$, we draw an independent uniformly random number $z_v \in [0,1]$. The state of node $v$ is

- $0$ if $Z_v = \sum_{i=1}^N \alpha_v^i = 0$,
- $i > 0$ if $z_v \in \left(\frac{\sum_{j=1}^{i-1} \alpha_v^j}{Z_v}, \frac{\sum_{j=1}^{i} \alpha_v^j}{Z_v}\right]$.

Similarly, for the updates in the diffusion stage for both $\mathcal{G}$ and $\hat{\mathcal{G}}$, an equivalent way of describing the update in step $t$ is the following. Draw an independent uniformly random number $z_t \in [0,1]$. If node $v_t$ is in a state $i > 0$, she retains her current state. Otherwise, she changes her state to

- $i > 0$ if $z_t \in [\sum_{j=1}^{i-1} h_{v_t}^j(\boldsymbol{S}_{t-1}), \sum_{j=1}^{i} h_{v_t}^j(\boldsymbol{S}_{t-1}))$,
- $0$ otherwise.

To couple the two random processes, we simply choose the same values $z_v = \hat{z}_{t_{v'}}$ and $z_t = \hat{z}_{t+|V|}$, where $t_{v'}$ is the order of node $v'$ in the update sequence $\hat{Q}$.

Since the strategy $\hat{\boldsymbol{M}}$ consists of sets instead of multisets, the seeding stage of $\hat{\mathcal{G}}$ is deterministic. In $\hat{\mathcal{G}}$, a node $v$ is initially in state $i > 0$ if and only if player $i$ selects her as a seed. Let $\hat{S}_0^i$ be the set of activated nodes after the seeding stage with strategy profile $\hat{\boldsymbol{M}}$. We have $|\hat{S}_0^i \cap V_v^i| = \alpha_v^i$, for all players $i$ and nodes $v$.

Conditioned on any fixed choice of the $z_v$ (and thus $\hat{z}_1, \ldots, \hat{z}_{|V|}$), we have $\boldsymbol{S_0} = \hat{\boldsymbol{S}}_{|V|}$, where $\boldsymbol{S_0}$ is the vector of sets of nodes in state $i$ after the seeding stage with strategy profile $\boldsymbol{M}$, and $\hat{\boldsymbol{S}}_{|V|}$ is the vector of sets of nodes in layer 2 of $\hat{\mathcal{G}}$ in state $i$ after the first $|V|$ update steps with strategy profile $\boldsymbol{M}$.

Finally, a simple induction proof over the $\ell$ steps of the update sequence $Q$ shows that for each time $t$, we have the following property: (1) if $v$ appears in $Q$ at least once before time step $t$, then $v \in S_t^i$ if and only if $v \in \hat{S}_{t+|V|}^i$. (2) if $v$ does not appear in $Q$ before time step $t$, then $v \in S_t^i$ if and only if $v' \in \hat{S}_{t+|V|}^i$. Applying this result after all $\ell$ steps, we obtain that each node $v$ appearing in $Q$ has $v \in S_\ell^i$ if and only if $v \in \hat{S}_{\ell+|V|}^i$, and each node $v$ not appearing in $Q$ has $v \in S_\ell^i$ if and only if $v' \in \hat{S}_{\ell+|V|}^i$. Notice that the corresponding nodes $v$ or $v'$ in $\hat{G}$ are exactly the ones inheriting the weight of node $v$ in $G$, implying that the payoff of each player $i$ is the same pointwise in $\mathcal{G}$ and $\hat{\mathcal{G}}$. Thus, each player's expected payoff is also the same in the two games, completing the proof. ■